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LETTER TO THE EDITOR

Transfer-matrix approach to the one-dimensional percolation problem

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Abstract. The transfer-matrix method is used to find the exact phase diagrams and the correlation length exponents ν for the one-dimensional site, bond and site–bond percolation problems with bonds connecting the Lth-nearest neighbours (L up to 3). For the site percolation, our results agree with the exact result obtained from the generating function method, $\nu = L$, while for the bond percolation, we found much richer critical phenomena. If all the L bonds have equal occupation probability, our results predict $\nu = L(L+1)/2$.

In the past decade, the percolation problem has caused renewed and wide interest because of its close relationship with thermal critical phenomena (Kasteleyn and Fortuin 1969). Recently, the generating function method has been used to solve the one-dimensional site percolation problem with Lth-nearest-neighbour bonds (Klein *et al* 1978). The critical behaviour is found to be L dependent. In particular, the correlation length exponent ν is found to be exactly equal to the number of neighbouring bonds L. However, if the same method is applied to the one-dimensional bond percolation with further-neighbour bonds, it will turn out to be rather complicated. It is well known that the one-dimensional Ising model with only nearest-neighbour interaction can be solved exactly by both the generating function and transfer-matrix methods. A recursive method which is similar to the transfer-matrix method was recently introduced to solve the one-dimensional Ising model with higher-order interaction (Marchi and Vila 1980). It will be interesting to apply the transfer-matrix method to the one-dimensional percolation problem with further-neighbour bonds. In this letter, we report some results obtained by using the transfer-matrix method to the one-dimensional site, bond and site–bond percolation problems with further-neighbour bonds (L up to 3).

Case (1): site percolation with $L = 2$

It is trivial to solve the one-dimensional percolation problem with only the nearest-neighbour bond. We will start with the next simplest case; site percolation with the next-nearest-neighbour bonds ($L = 2$). In this case, the one-dimensional chain can be drawn schematically in figure 1. The $(i, i+1)$ and $(i, i+2)$ bonds are respectively the nearest- and next-nearest-neighbour bonds.

In the standard transfer-matrix method, one usually partitions the system into columns, and the transfer-matrix transfers the probability distribution of various configurations in the Nth column to the $(N+1)$th column. If we assume that the probability of the first and Nth columns being connected decays as $\exp(-N/\xi)$ for large N, where ξ is the correlation length, then ξ can be obtained from the largest eigenvalue λ_m (excluding the trivial value $\lambda = 1$) of the transfer matrix by using the relation (Derrida and Vannimenus 1980)

$$\xi = -1/\ln \lambda_m. \tag{1}$$

In figure 1, suppose we take the sites i and $i+1$ as the Nth column; then we can choose either the sites $i+1$ and $i+2$ as the next column or the sites $i+2$ and $i+3$ as the next column. Both ways will lead to exactly the same critical behaviour. Here we will choose the former one because it gives a much simpler transfer matrix. By doing so, the word 'column' does not have any geometric meaning. However, we will still use the name for convenience.

There are four possible configurations in each column. For the Nth column, we have the following configurations: (1) neither i nor $i+1$ connected to the first column, (2) i but not $i+1$ connected, (3) $i+1$ but not i connected, (4) both i and $i+1$ connected. Similarly there are four configurations in the $(N+1)$th column with i and $i+1$ replaced by $i+1$ and $i+2$ respectively. If $P_n(N)$ is the probability that the Nth column is in the configuration n, then the transfer matrix $M^{(s2)}_{mn}$, which connects $P_m(N+1)$ and $P_n(N)$, is

$$P_m(N+1) = \sum_{n=1}^{4} M^{(s2)}_{mn} P_n(N), \tag{2}$$

$$M^{(s2)}_{mn} = \begin{bmatrix} 1 & q & 0 & 0 \\ 0 & 0 & q & q \\ 0 & p & 0 & 0 \\ 0 & 0 & p & p \end{bmatrix}, \tag{3}$$

where p is the site occupation probability and $q = 1-p$. The superscript '(s2)' of M denotes the transfer matrix for site percolation with $L = 2$. It is easy to find that $M^{(s2)}$ has eigenvalues $\lambda = 1, 0, [p \pm (p^2+4pq)^{1/2}]/2$. Since the correlation length ξ is infinite at the critical percolation p_c, from (1) it follows that $\lambda_m(p_c) = 1$. Here we have $\lambda_m = [p+(p^2+4pq)^{1/2}]/2$ which gives $p_c = 1$. In the critical region, q is small ($q = 1 - p = p_c - p$). Expanding λ_m for small q, to the leading order in q we find $\lambda_m = 1 - q^2 + o(q^3)$. Substituting λ_m into (1), we have

$$\lim_{q\to 0} \xi = -\frac{1}{\ln(1-q^2)} \approx q^{-2}. \tag{4}$$

From (4) we obtain $\nu = 2$, which agrees with the exact results of Klein *et al* (1978).

Case (2): site percolation with $L = 3$

For $L = 3$, we drew a similar schematic diagram for the one-dimensional chain in figure 2, where the bond connecting sites i and $i+3$ is a third-neighbour bond. Now each column contains three sites. We choose the sites i, $i+1$ and $i+2$ as the Nth

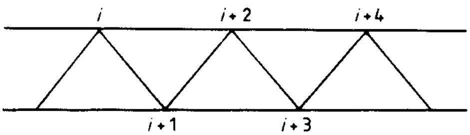

Figure 1. Schematic diagram for a one-dimensional chain with next-nearest-neighbour bonds.

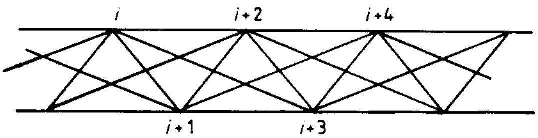

Figure 2. Schematic diagram for a one-dimensional chain with third-nearest-neighbour bonds.

column and the sites $i+1$, $i+2$ and $i+3$ as the $(N+1)$th column. There are eight configurations in each column. For the Nth column, we have the following configurations: (1) none of i, $i+1$ and $i+2$ connected to the first column, (2) i but not $i+1$ or $i+2$ connected, (3) $i+1$ but not i or $i+2$ connected, (4) $i+2$ but not i or $i+1$ connected, (5) both i and $i+1$ but not $i+2$ connected, (6) both i and $i+2$ but not $i+1$ connected, (7) both $i+1$ and $i+2$ but not i connected, (8) all of i, $i+1$ and $i+2$ connected. If $W_n(N)$ is the probability that the Nth column is in the configuration n, we have the following transfer matrix:

$$W_m(N+1)=\sum_{n=1}^{8} M_{mn}^{(s3)} W_n(N), \tag{5}$$

$$M_{mn}^{(s3)}=\begin{bmatrix} 1 & q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & q & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & q \\ 0 & 0 & p & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p & p \end{bmatrix}. \tag{6}$$

It is easy to show that the above transfer matrix has maximum non-trivial eigenvalue in the small q limit, $\lambda_m = 1-q^3+o(q^4)$. From (1), we obtain $p_c=1$, $\lim_{q\to 0}\xi = q^{-3}$ and $\nu=3$. This again agrees with the exact results of Klein *et al* (1978). The transfer matrix has the dimension $2^L\times 2^L$, and becomes increasingly difficult to solve when L becomes large. Since there are no approximations in this method, we find no reasons to doubt that this method will also yield the exact result: $\nu=L$ for any finite L.

Case (3): bond percolation with $L=2$

Let p and r respectively be the occupation probabilities for the nearest- and next-nearest-neighbour bonds and $q=1-p$, $s=1-r$. Using the same definition for $P_n(N)$ as in case (1), we now have a different transfer matrix

$$P_m(N+1)=\sum_{n=1}^{4} M_{mn}^{(b2)} P_n(N), \tag{7}$$

$$M_{mn}^{(b2)}=\begin{bmatrix} 1 & s & 0 & 0 \\ 0 & 0 & q & sq \\ 0 & r & 0 & 0 \\ 0 & 0 & p & 1-sq \end{bmatrix}. \tag{8}$$

From the maximum non-trivial eigenvalue of (8) we find the following results (cf figure 3). (i) Lines AC and BC are the only two critical lines on which $\lambda_m=1$ and ξ is infinite. (ii) Near the line CB (excluding point C), q is small. We have $\lim_{q\to 0}\xi = s^{-2}q^{-1}$ and $\nu=1$. (iii) Near the line AC (excluding points A and C), s is small. We have $\lim_{s\to 0}\xi = pq^{-1}s^{-2}$ and $\nu=2$. (iv) Near the point A but on the line OA, we have $\lim_{s\to 0}\xi = 2s^{-1}$ and $\nu=1$. These are just the results of the pure nearest-neighbour bond but with twice the bond length. (v) Near the point C, both q and s are small. We have $\lim_{q,s\to 0}\xi = q^{-1}s^{-2}$ and $\nu=3$. Here the limit that both q and s approach zero is taken simultaneously. We can also put $q=cs$ and then take the $s\to 0$ limit, where c is any positive constant.

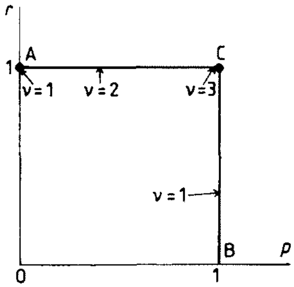

Figure 3. Phase diagram and correlation length exponent for case (3).

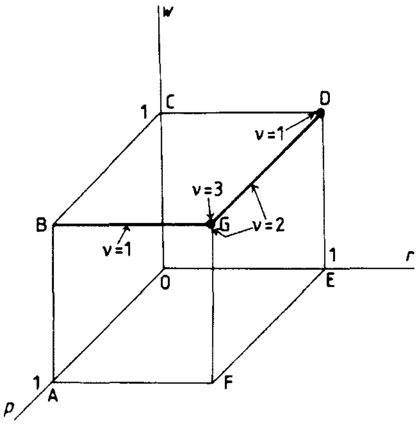

Figure 4. Phase diagram and correlation length exponent for case (4).

Case (4): site–bond percolation with $L = 2$

In addition to case (3), now we also dilute the site occupation. Let w be the site occupation probability with $u = 1 - w$; then the transfer matrix (8) becomes

$$M_{mn}^{(\mathrm{sb2})} = \begin{bmatrix} 1 & 1-rw & 0 & 0 \\ 0 & 0 & 1-pw & 1-w+wsq \\ 0 & rw & 0 & 0 \\ 0 & 0 & pw & w(1-sq) \end{bmatrix}. \tag{9}$$

The above matrix reduces to (8) of case (3) when $w = 1$, and reduces to case (1) when $p = r = 1$. From (9) we find the following results (cf figure 4). (i) Lines BG and GD are the only critical lines. (ii) Near the line BG (excluding point G), u and q are small. We have $\lim_{q,u\to 0} \xi = (su + s^2q)^{-1}$ and $\nu = 1$. (iii) Near the line DG (excluding the points D and G), s and u are small. We have $\lim_{s,u\to 0} \xi = p((1+q)su + u^2 + qs^2)^{-1}$ and $\nu = 2$. (iv) Near the point D but on the $p = 0$ plane, we have $\lim_{s,u\to 0} \xi = 2(s+u)^{-1}$ and $\nu = 1$. (v) Near the point G, if we approach the point G from the $w = 1$ plane, we have, for small s and q, $\lim_{s,q\to 0} \xi = q^{-1}s^{-2}$ and $\nu = 3$. (vi) Near the point G but approaching G from below the $w = 1$ plane, q, s and u are all small. We have $\lim_{q,s,u\to 0} \xi = (su + u^2)^{-1}$ and $\nu = 2$.

Case (5): bond percolation with $L = 3$

Let p, r and x be the occupation probabilities for the nearest, next-nearest and third-nearest-neighbour bonds with $q = 1 - p$, $s = 1 - r$ and $y = 1 - x$. Using the same definition for $W_n(N)$ as in case (2), instead of (6), the transfer matrix becomes

$$M_{mn}^{(\mathrm{b3})} = \begin{bmatrix} 1 & y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & sy & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & qy & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & qs & qsy \\ 0 & 0 & r & 0 & 1-sy & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & 1-qy & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-qs & 1-qsy \end{bmatrix} \tag{10}$$

From (10), we find the following results (cf figure 5). (i) $p=1$, $r=1$ and $x=1$ are the only three critical surfaces. (ii) Near the $p=1$ surface (excluding lines BG and FG), q is small. We have $\lim_{q\to 0}\xi = s^{-2}y^{-3}q^{-1}$ and $\nu=1$. (iii) Near the $r=1$ surface (excluding point E and lines DG and FG), s is small. We have $\lim_{s\to 0}\xi = (1-yq)(1+qx)^{-1}q^{-1}y^{-3}s^{-2}$ and $\nu=2$. (iv) Near the $x=1$ surface (excluding point C and lines BG and DG), y is small. We have $\lim_{y\to 0}\xi=(1-sq)^2(1+q)^{-1}q^{-1}s^{-2}y^{-3}$ and $\nu=3$. (v) Near the line BG (excluding point G), both q and y are small. We have $\lim_{q,y\to 0}\xi = s^{-2}y^{-3}q^{-1}$ and $\nu=4$. (vi) Near the line FG (excluding point G), both q and s are small. We have $\lim_{q,s\to 0}\xi = y^{-3}q^{-1}s^{-2}$ and $\nu=3$. (vii) Near point E but on the OE line, s is small. We have $\lim_{s\to 0}\xi = 2s^{-1}$ and $\nu=1$. (viii) Near the line DG (excluding point G), both s and y are small. We have $\lim_{s,y\to 0}\xi = (1+q)^{-1}q^{-1}s^{-2}y^{-3}$ and $\nu=5$. (ix) Near point C but on the OC line, y is small. We have $\lim_{y\to 0}\xi = 3y^{-1}$ and $\nu=1$. This again is consistent with the results of the pure nearest-neighbour bond with three times the bond length. (x) Near point G, all of q, s and y are small. We have $\lim_{q,s,y\to 0}\xi = q^{-1}s^{-2}y^{-3}$ and $\nu=6$.

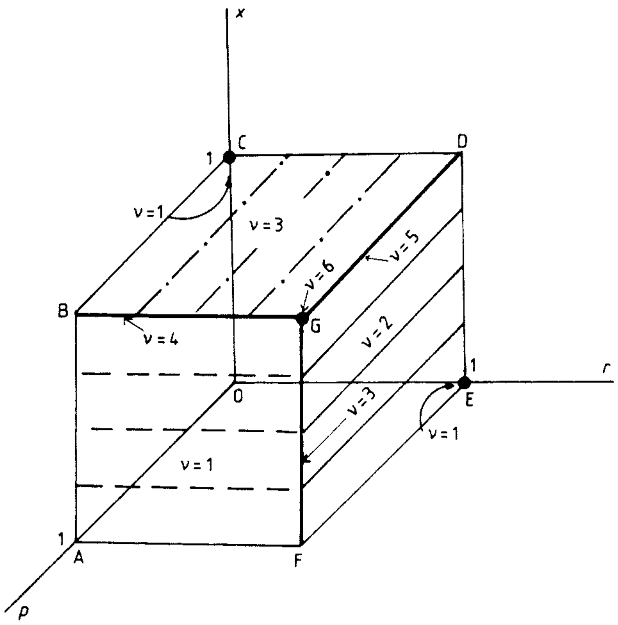

Figure 5. Phase diagram and correlation length exponent for case (5).

Comparing the results of different cases discussed above, it is possible to predict the phase diagrams and much critical behaviour for more complicated systems with higher-neighbour bonds. For instance, if we let all of the L neighbouring bonds have the same occupation probability ($p=r=x=\ldots$), through comparing the results (v) of case (3) and (x) of case (5), it is plausible to predict that the critical exponent for such a system will be $\nu=1+2+3+\ldots+L=L(L+1)/2$.

In summary, we have used the transfer-matrix method to find the exact critical behaviour for the one-dimensional site, bond and site–bond percolation problems with bonds connecting Lth-nearest neighbours (L up to 3). For the site percolation, our results agree with the exact results obtained from the generating function method.

We find that the bond percolation contains much richer critical phenomena than the site percolation, especially when L becomes large. If all the L bonds have equal occupation probability, we predict that the critical exponents will be $\nu = L(L+1)/2$.

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